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Note

M_2-equipackable graphs☆

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Abstract

Let H be a fixed graph. An H-*packing* of G is a set of edge disjoint subgraphs of G each isomorphic to H. An H-packing in G with k copies $H_1, H_2, \ldots, H_k$ of H is called *maximal* if $G - \bigcup_{i=1}^{k} E(H_i)$ contains no subgraph isomorphic to H. An H-packing in G with k copies $H_1, H_2, \ldots, H_k$ of H is called *maximum* if no more than k edge disjoint copies of H can be packed into G. A graph G is called H-*equipackable* if every maximal H-packing in G is also a maximum H-packing in G. By M_t, $t \geqslant 1$, we denote a matching having t edges. In this paper, we investigate the characterization of M_2-equipackable graphs.

Keywords: Packing; Packable; Randomly packable; Equipackable; Matching

1. Preliminaries and known results

A vertex of degree 0 is called an isolated vertex (or an isolate). The graphs characterized in this paper do not have any isolated vertices. A graph G has order $|V(G)|$ and size $|E(G)|$. The path and circuit on k vertices are denoted by P_k and C_k, respectively. A matching in the graph G is a set of independent edges in G. By M_t, $t \geqslant 1$, we denote a matching having t edges. Let H be a subgraph of G. By $G - H$, we denote the graph remaining after we delete from G the edges of H and any resulting isolated vertices. A collection of edge disjoint copies of H, say $H_1, H_2, \ldots, H_k$, where each H_i is a subgraph of G, is called an H-*packing* in G. An H-packing in G with k copies $H_1, H_2, \ldots, H_k$ of H is called *maximal* if $G - \bigcup_{i=1}^{k} E(H_i)$ contains no subgraph isomorphic to H. An H-packing in G with k copies $H_1, H_2, \ldots, H_k$ of H is called *maximum* if no more than k edge disjoint copies of H can be packed into G. A graph G is called H-*packable* if there exists an H-packing of G which uses all edges in G and G is called *randomly* H-*packable* if every maximal H-packing in G uses all edges in G. There have been many results on H-packable graphs and randomly H-packable graphs (see [1,3]). The following results when $H = M_2$ are very useful to our work:

Theorem 1.1 (*Caro [4]*). *Let G be a graph of size $2m > 0$ and without isolated vertices. Then G is M_2-packable if and only if $\Delta(G) \leqslant m$ and G is not isomorphic to $K_3 \cup K_2$.*

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Theorem 1.2 (*Ruiz [4]*). *Let G be a graph of size* $2m>0$ *without isolated vertices and suppose G is not isomorphic to* M_2, *then G is randomly* M_2*-packable if and only if* $G \in \mathscr{F}$, *where* $\mathscr{F}=\{K_4, C_4, 2K_3, K_3 \cup K_{1,3}\} \cup \{2mK_2, 2K_{1,m} | m \geqslant 2\}$.

As a relaxation of random H-packability, Hartnell and Vestergaard [2] introduced the definition of H-equipackable. A graph G is called H*-equipackable* if every maximal H-packing in G is also a maximum H-packing in G. Hartnell and Vestergaard [2] first characterized the P_3-equipackable graphs of girth five or more. Later, Vestergaard [6] gave a characterization of P_3-equipackable graphs with all valences at least two. Recently, Vestergaard and a coauthor have extended the results of [2,6] by characterizing all P_3- equipackable graphs without any restrictions on the graph.

In this paper, we investigate M_2-equipackable graphs.

Note that when G is congruent to K_2 or P_3 there does not exist a copy of M_2. When $G \cong M_2$, G is clearly M_2-equipackable. So, in the sections that follow, we consider graphs with size at least 3.

2. M_2-equipackable graphs with even size

Lemma 2.1. *Let G be a graph with size* $2m$ *and maximum degree d. If* $d>m$, *then the number of* M_2 *in the maximum* M_2*-packing of G is* $2m-d$.

Proof. Assume that v is a vertex with degree d and $v_1, v_2, \ldots, v_d$ are the neighbors of v . Let $E_1=\{vv_i | i=1,2,\ldots,d\}$ and $E_2=E(G)-E_1=\{e_1,e_2,\ldots,e_{2m-d}\}$. Since d and m are both positive integers and $d>m$, $|E_1|-|E_2|=d-(2m-d)=2(d-m)\geqslant 2$. Clearly, each edge of E_2 is adjacent to at most two edges of E_1, so for any $e_i \in E_2$, there exists an edge $vv_j \in E_1$ such that $\{e_i, vv_j\}$ forms a copy of M_2. Remove $\{e_i, vv_j\}$. Let $E_1^{(1)}=E_1-\{vv_j\}$ and $E_2^{(1)}=E_2-\{e_i\}$. In the same way, each edge in $E_2^{(1)}$ has at most two neighbors in $E_1^{(1)}$, then we can get another copy of M_2 and remove it. Repeating this process $2m-d$ times, we can remove all the edges of E_2, each of which along with one edge of E_1 forms a copy of M_2. And the remaining edges in E_1 contains no M_2. So the removed $2m-d$ copies of M_2 form a maximal M_2-packing of G. Because all edges in E_1 are adjacent and no two edges in E_2 belong to one copy of M_2 at the same time, this M_2-packing is maximum. □

Theorem 2.2. *Let G be a graph with size* $2m>2$, *then G is* M_2*-equipackable if and only if G satisfies one of the following*:

(1) $G \cong K_3 \cup K_2$;
(2) $G \in \mathscr{F}$, *where* $\mathscr{F}=\{K_4, C_4, 2K_3, K_3 \cup K_{1,3}\} \cup \{2mK_2, 2K_{1,m} | m \geqslant 2\}$;
(3) $\Delta(G)=d>m$, *and for any vertex v whose degree is d, the subgraph induced by* $E(G-v)$ *must be* $K_{1,2m-d}$ *or* K_3.

Proof. We can easily verify that the graphs described in (1)–(3) are all M_2-equipackable.

Conversely, let G be an M_2-equipackable graph with size $2m$. We consider two cases:

Case 1: $\Delta(G) \leqslant m$.

Subcase 1: When $G \cong K_3 \cup K_2$, G is clearly M_2-equipackable.

Subcase 2: When G is not isomorphic to $K_3 \cup K_2$, we have:

By Theorem 1.1, G is M_2-packable. So the number of M_2 in the maximum M_2-packing is m. If G is not randomly M_2-packable, then there exists a maximal M_2-packing which does not use all edges in G and, consequently, which is not maximum. This contradicts the fact that G is M_2-equipackable. So G must be randomly M_2-packable. By Theorem 1.2, $G \in \mathscr{F}=\{K_4, C_4, 2K_3, K_3 \cup K_{1,3}\} \cup \{2mK_2, 2K_{1,m} | m \geqslant 2\}$.

Case 2: $\Delta(G)=d>m$.

By Lemma 2.1, the number of M_2 in the maximum M_2-packing in G is $2m-d$. Let v be a vertex with maximum degree. If there are two edges (say $\{e,f\}$) in $G-v$ which are not adjacent, then after removing $\{e,f\}$ (we denote $G-\{e,f\}$ by G_1), $\Delta(G_1)=d>m>m-1$. The graph G_1 also satisfies Lemma 2.1. So we can get a maximum M_2-packing in G_1 with $2(m-1)-d$ copies of M_2 which, along with $\{e,f\}$, form a maximal M_2-packing. Obviously this resulting maximal packing with only $2m-d-1$ copies of M_2 is not maximum, which contradicts that G is

M_2-equipackable. So all edges in $G - v$ are mutually adjacent; that is, the subgraph induced by $E(G - v)$ must be $K_{1,2m-d}$ or K_3. □

3. M_2-equipackable graphs with odd size

The following lemma can be proven in a manner similar to Lemma 2.1. The proof is thus omitted.

Lemma 3.1. *Let G be a graph with size* $2m + 1$. *If* $\Delta(G) = d > m + 1$, *then the number of* M_2 *in the maximum* M_2*-packing of G is* $2m + 1 - d$.

Theorem 3.2. *Let G be a graph with size* $2m + 1$ *and* $\Delta(G) = d \geqslant m + 2$, *then G is* M_2*-equipackable if and only if, for any vertex v whose degree is d, the subgraph induced by* $E(G - v)$ *must be* $K_{1,2m+1-d}$ *or* K_3.

Proof. Applying the same technique as in the proof of Case 2 of Theorem 2.2, we can easily get the result. □

Lemma 3.3. *Let G be a graph with size* $2m + 1$ *and F be a maximal* M_2*-packing which omits all the edges of a star* $K_{1,3}$. *If F contains a copy (say* $\{e, f\}$*) of* M_2 *such that neither e nor f is incident with the center of the star* $K_{1,3}$, *then G is not* M_2*-equipackable.*

Proof. Without loss of generality, we denote the edges of the star $K_{1,3}$ by vv_1, vv_2, vv_3. Neither e nor f is incident with v. For the edge e, there are three cases:

Case 1: The edge e is not incident with any vertex of $\{v_1, v_2, v_3\}$. Since there are at most two vertices of $\{v_1, v_2, v_3\}$ which are incident with f, say v_1 and v_2, we can replace $\{e, f\}$ with $\{vv_3, f\}$ and $\{e, vv_1\}$ to get a maximum M_2-packing whose size is larger than that of the given maximal M_2-packing. (See Fig. 1 (a).) So G is not M_2-equipackable.

Case 2: The edge e is incident with just one vertex of $\{v_1, v_2, v_3\}$, say v_1. Then f is not incident with v_1 since e and f are independent. We can replace $\{e, f\}$ with $\{vv_1, f\}$ and $\{e, vv_3\}$ to get another larger maximal M_2-packing which is maximum. (See Fig. 1 (b).) So G is not M_2-equipackable.

Case 3: The edge e is incident with two vertices of $\{v_1, v_2, v_3\}$, say v_1 and v_2. Then neither v_1 nor v_2 is incident with f since e and f are independent. We can replace $\{e, f\}$ with $\{vv_1, f\}$ and $\{e, vv_3\}$ to get another larger maximal M_2-packing. (See Fig. 1 (c).) So G is not M_2-equipackable.

In all cases, G is not M_2-packable. □

Lemma 3.4. *Let G be a graph with size* $2m+1$ *and maximum degree d*, $2 < d < m+2$. *If G is* M_2*-equipackable, then for any subgraph H of G which is isomorphic to* $K_{1,3}$, $G - H$ *is not* M_2*-packable.*

Proof. Assume that the lemma is not true; that is, $G - H$ is M_2-packable. So $G - H$ can be the union of $(m - 1)$ copies of M_2. There exists a copy (say $\{e, f\}$) of M_2 in $G - H$ such that neither e nor f is incident with the center of the star H. (Otherwise, each copy of M_2 in $G - H$ has an edge incident with the center (say v) of H, and then $d = \Delta(G) \geqslant d_G(v) \geqslant m - 1 + 3 = m + 2$, which contradicts that $d < m + 2$.) By Lemma 3.3, G is not M_2-equipackable. This is a contradiction. □

Lemma 3.5. *Let G be a graph with size* $2m + 1$. *If there is a maximal* M_2*-packing of G which omits all the edges of a subgraph* K_3, *then G is not* M_2*-equipackable.*

Proof. Without loss of generality, we denote by a, b, c the edges of the subgraph K_3. Select a copy of M_2, say $\{e, f\}$. In the given maximal M_2-packing of G, for $\{e, f\}$, there are three cases on the adjacency relation with $\{a, b, c\}$:

Case 1: There is no edge in $\{e, f\}$ which is adjacent to any edge of $\{a, b, c\}$. (See Fig. 2 (a).) We can replace $\{e, f\}$ with $\{e, a\}$ and $\{b, f\}$ to get another M_2-packing which is maximum. So G is not M_2-equipackable.

Case 2: Only one edge of $\{e, f\}$, say e, is adjacent to two edges, say a, c, of $\{a, b, c\}$. (See Fig. 2 (b).) Then we can replace $\{e, f\}$ with $\{e, b\}$ and $\{a, f\}$ to get another M_2-packing which is maximum. So the given M_2-packing is not maximum and G is not M_2-equipackable.

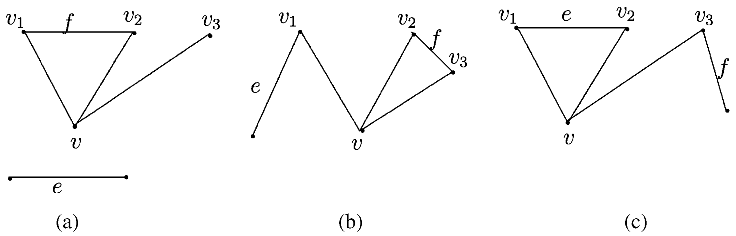

Fig. 1. All cases on the incident relation of e and $\{v_1, v_2, v_3\}$.

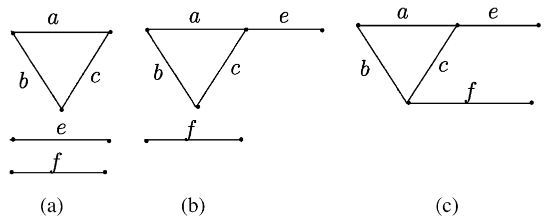

Fig. 2. All cases on the adjacency relation of $\{e, f\}$ and $\{a, b, c\}$.

Case 3: Both of the edges in $\{e, f\}$ are adjacent to the edges of $\{a, b, c\}$. Without loss of generality, suppose that e is adjacent to a, c and f is adjacent to b, c. (See Fig. 2 (c).) We can replace $\{e, f\}$ with $\{e, b\}$ and $\{a, f\}$ to get another M_2-packing which is maximum. Thus G is not M_2-equipackable. Thus, in all cases, G is not M_2-packable. □

The following corollary easily follows from Lemma 3.5:

Corollary 3.6. *Let G be a graph with size* $2m+1$. *If G is* M_2*-equipackable, then for any subgraph which is isomorphic to* K_3, $G - H$ *is not* M_2*-packable.*

Theorem 3.7. *Let G be a graph with size* $2m + 1 > 1$ *and* $\Delta(G) = d < m + 2$, *then G is equipackable if and only if G satisfies one of the following*:

(1) *Neither* K_3 *nor* $K_{1,3}$ *is contained in G as a subgraph.*
(2) *At least one copy of* K_3 *or* $K_{1,3}$ *is contained in G as a subgraph and, for any subgraph H of G which is isomorphic to* K_3 *or* $K_{1,3}$, $\Delta(G - H) > m - 1$ *or* $G - H \cong K_3 \cup K_2$.

Proof. We can easily verify that the graphs described in the theorem are all M_2-equipackable.

Conversely, let G be M_2-equipackable. We consider two cases:

Case 1: Neither K_3 nor $K_{1,3}$ is contained in G as a subgraph. Clearly, G is M_2-equipackable.

Case 2: At least one copy of K_3 or $K_{1,3}$ is contained in G as a subgraph. By Lemma 3.4 and Corollary 3.6, for any subgraph H which is isomorphic to K_3 or $K_{1,3}$, $G - H$ is not M_2-packable so, by Theorem 1.1, $\Delta(G - H) > m - 1$ or $G - H \cong K_3 \cup K_2$. □

Especially, we have:

Corollary 3.8. *Odd cycles and odd paths are all* M_2*-equipackable.*

As far as we know, only H-equipackable graphs for P_3 and M_2 have been researched. One might consider H-equipackable graphs for other graphs such as P_k $(k = 4, 5, 6)$, M_t $(t \geqslant 3)$, K_n and $K_{1,r}$.

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